Coherent Sheaves

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References are [GW20], [Ris],https://www.math.uni-bonn.de/people/ja/alggeoI/notes.pdf

1 The Category of Ringed Spaces

A ringed space is simply a pair $(X, \mathcal{O}_X)$ consisting of a topological space and a sheaf of rings. A morphism of ringed spaces

$$(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a continuous function $f : X \to Y$ and a morphism of sheaves $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$, to the push forward of $\mathcal{O}_X$ by f, $f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$. We need to pushforward, otherwise we would have two sheaves on different topological spaces and we wouldnt know how to talk about morphisms between them.

Example (Manifolds)**.** *If we are given a (smooth) manifold $\mathcal{M}$ then we can consider the sheaf of smooth functions $\mathcal{E}$. This is a sheaf of rings, even $\mathbb{R}$-algebras.*

If $(\mathcal{M}_1, \mathcal{E}_1)$ and $(\mathcal{M}_2, \mathcal{E}_2)$ are two ringed spaces given by smooth functions on smooth manifolds, then pre-composition gives a morphism between them

$$\begin{array}{ccc} \mathcal{E}_2(V) & \xrightarrow{(-)\circ f} & \mathcal{E}_1(U) \\ \Big\downarrow{\scriptstyle res^V_{V'}} & & \Big\downarrow{\scriptstyle res^U_{U'}} \\ \mathcal{E}_2(V') & \xrightarrow{(-)\circ f} & \mathcal{E}_1(U') \end{array}$$

for every collection of opens $U' \subseteq U \subseteq \mathcal{M}_1, V' \subseteq V \subseteq \mathcal{M}_2$ such that $f(U) \subseteq V, f(U') \subseteq V'$.

Example (Schemes)**.** *Given a ring A then we define the a topological space* $\mathrm{Spec}(A) := \{\mathfrak{p} : \mathfrak{p} \subseteq A \text{ a prime ideal}\}$*, with open sets generated (this is a basis) by the distinguished opens for $f \in A$*

$$D(f) := \{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}$$

the sheaf of "regular functions" is defined on this distinguished basis by

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) \quad := \quad A_f = \{1, f, f^2, ...\}^{-1}A$$

This is a ringed space.

Remark. *We have not defined affine schemes or schemes because we havent defined* ***locally*** *ringed spaces. The definition is an extra condition on ringed space, and the morphisms respect this structure. Then affine schemes are locally ringed spaces isomorphic to the spectrum of a ring and schemes are locally ringed spaces that are locally affine schemes.*

We have not explained ***why*** *these definitions might make sense to make. To do that one must actually do some work in say geometry.*

2 The Category of $\mathcal{O}_X$ Modules

Now we fix a ringed space $(X, \mathcal{O}_X)$. An $\mathcal{O}_X$-module $\mathscr{F}$ is then a sheaf of $\mathbb{Z}$ -modules (abelian groups) such that for every $U \subseteq X$ open $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$ module and the action commutes with restriction maps.

We define a morphism of $\mathcal{O}_X$-modules $\mathscr{F} \to \mathscr{G}$ as a morphism of sheaves that moreover gives a morphism of $\mathcal{O}_X(U)$ modules $\mathscr{F}(U) \to \mathscr{G}(U)$ for every open $U \subseteq X$.

An $\mathcal{O}_X$-module is of finite type if around every point there is a neighbourhood U such that $\mathscr{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$ module.

Example (Vector Bundle)**.** *Consider again the sheaf of smooth functions on a smooth manifold $(\mathcal{M}, \mathcal{E})$. Let $\pi : V \to X$ be a vector bundle.*

We define another sheaf on X by

$$\mathcal{S}(U) \quad := \quad \{\sigma : U \to V | \sigma \text{ is a smooth section of } V\}$$

We can see that we can add sections pointwise

$$(\sigma + \sigma')(u) = \sigma(u) + \sigma'(u)$$

because of the vector space structure and so indeed this is a sheaf of abelian groups. Moreover it is clear that if $f : U \to \mathbb{R}$ is a smooth function (i.e. $f \in \mathcal{E}(U)$) then

$$f.\sigma(u) = f(u)\sigma(u)$$

is another section of V, and that this commutes with restriction. Hence the sheaf of sections of a vector bundle is an $\mathcal{E}$ module.

Example ($\underline{\mathbb{Z}}$ and abelian groups)**.** *If $\underline{\mathbb{Z}}$ is the constant sheaf assigning to every open $\mathbb{Z}$ then a $\underline{\mathbb{Z}}$-module is exactly a sheaf of abelian groups.*

Example (Modules on the Point)**.** *If the topological space is $X = \{*\}$ and as a ringed space $\mathcal{O}_X(*) = A$ then a $\mathcal{O}_X$-module is just an A module.*

Let $\square$ be a construction of modules, $\square : \prod_i Mod_R \to Mod_R$, then the construction of sheaves,$\square^{sh} : \prod_i \mathcal{O}_X - mod \to \mathcal{O}_X - mod$, is given by defining a presheaf on opens $\mathscr{F}(U) \quad := \quad \square(\mathscr{G}_i(U))_i$ and then sheafifying i.e.

$$\square^{sh} = \mathscr{F}^*$$

Note that the sheafification is sometimes superfluous.

Example (Direct Sum)**.** *Let $\mathscr{F}, \mathscr{G}$ be two $\mathcal{O}_X$ modules, then we define $\mathscr{F} \oplus \mathscr{G}$ to be the sheafification of the presheaf defined by*

$$U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$$

3 Free, Locally Free, Coherent and Quasi-Coherent Sheaves

3.1 Locally Free

We are still in the setting of $(X, \mathcal{O}_X)$ being some ringed space. We say that an $\mathcal{O}_X$-module $\mathscr{F}$ is locally free if for every $x \in X$ there is some neighbourhood $x \in U \subseteq X$ such that $\mathscr{F}|_U$ is isomorphic (as an $\mathcal{O}_X$-module) to $\oplus_{i \in I} \mathcal{O}_X|_U$, note that I can depend on x. If I is (always) finite then we call this a locally free sheaf of finite type or finite locally free. The cardinality of this I is a function in x and this function is called the rank of $\mathscr{F}$. It is obvious that this rank is a locally constant function. A rank 1 locally free sheaf is called an invertible sheaf, a line bundle or perhaps a line sheaf.

Example (Vector Bundles)**.** *Return to our example of the sheaf of smooth functions on a smooth manifold $(\mathcal{M}, \mathcal{E})$. With a given rank n vector bundle $\pi : V \to X$ and the associated$\mathcal{E}$-module $\mathcal{S}$.*

Now around every point the bundle is locally trivial, i.e. there is a neighbourhood $U \subseteq \mathcal{M}$ such that $V \cong U \times \mathbb{R}^n$. Thus a section on this U can be identified with a map $\sigma : U \to U \times \mathbb{R}^n$ i.e. (because it must be the identity on U) n functions into $\mathbb{R}$. This is precisely an element of $\mathcal{E}^n(U)$.

Remark. *Vakil claims these are the same data, you can construct a vector bundle out of a locally free sheaf, the precide conditions under which this is true allude me, moreover he talks about transition functions and I dont see the relation.*

This is not an abelian category; in general it lacks kernels and cokernels.

Example (Vakil, 13.1.9)**.**

huh

3.2 Quasi-Coherent

An $\mathcal{O}_X$-module $\mathscr{F}$ is called quasi-coherent if for every $x \in X$ there is some neighbourhood $x \in U \subseteq X$ such that there exists some exact sequence of $\mathcal{O}_X|_U$-modules

$$\bigoplus_{j \in J} \mathcal{O}_X|_U \to \bigoplus_{i \in I} \mathcal{O}_X|_U \to \mathscr{F}|_U \to 0$$

where I and J can depend on x; that is we are are adding in cokernels to the category of locally free sheaves. Note that this is still not always an abelian subcategory of $\mathcal{O}_X$-modules, which makes sense because a priori we still havent added in kernels.

It doesnt seem like there are any good examples of this failure though

Lemma.

$$\textit{Locally Free} \implies \textit{Quasi-Coherent}$$

Proof. By definition of locally free we have an exact sequence

$$0 \to \oplus_i \mathcal{O}_X|_U \to \mathscr{F}|_U \to 0$$

Remark. *One motivation is adding in cokernels and making the category of vector bundles bigger, to help us do homological algebra. I only mention that there is a construction similar to* Spec *that takes in an A-module and gives a* Spec(A)*-module, the image of this functor is exactly quasi-coherent sheaves.*

Positive examples of coherence and quasi-coherence might not be easy without first doing the scheme stuff.

3.3 Coherent

An $\mathcal{O}_X$-module $\mathscr{F}$ is called coherent if it is of finite type and for every $U \subseteq X$ and $n \geq 0$ then every homomorphism $\omega : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$ has a kernel of finite type.

Lemma ([GW20][7.28)**.** *]*

$$\textit{Coherent} \implies \textit{Quasi-Coherent}$$

These modules form for arbitrary ringed spaces an abelian subcategory of $\mathcal{O}_X$-modules. However we shall see that depending on the structure of the ringed space it may be a more or less useful one.

Lemma. *Locally free of finite rank* $\implies$ *coherent iff* $\mathcal{O}_X$ *is coherent (over itself).*

Proof. The forward implication is immediate. The backwards implication follows from the fact that the direct sum of coherent modules is coherent.

Example (Coherent $\mathcal{E}$ Sheaves)**.** *Consider* $\mathbb{R}$ *as a smooth manifold, with the sheaf of smooth functions* $\mathcal{E}$*. We claim that this is not a coherent module (over itself).*

It is clearly of finite type. So to see that it is not coherent it is sufficient to find a sheaf morphism $\mathcal{E} \to \mathcal{E}$ *that has a kernel that is not of finite type. Consider the smooth function*

$$f(x) \quad := \quad \begin{cases} 0, & x \leq 0 \\ \exp(\frac{-1}{x^2}), & x > 0 \end{cases}$$

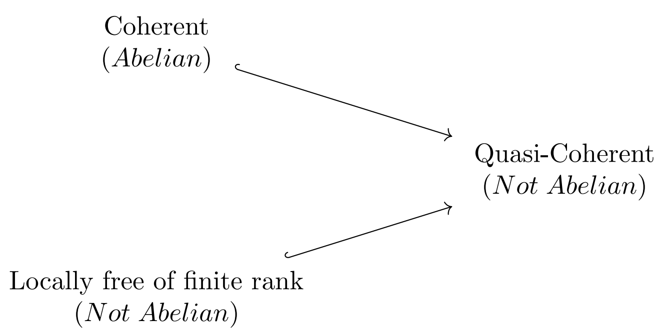
then multiplication by f *defines a sheaf morphism* $\mathcal{E} \to \mathcal{E}$*. The kernel is then the sheaf of functions* g *such that* $g(x) = 0, x > 0$ *which is not a sheaf of finite type.*

proof

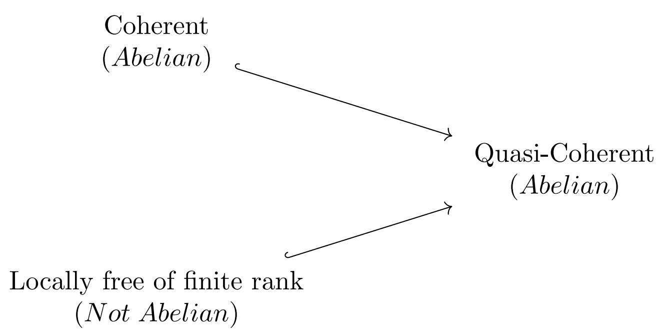
`https://math.stackexchange.com/questions/2163625/quasicoherent-sheaves-on-smooth-manifolds-and-their-applications?rq=1`

3.4 Compare

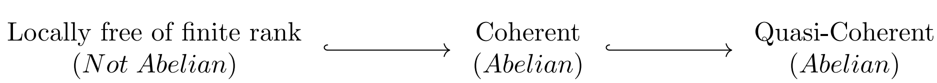
So in the general setting of $\mathcal{O}_X$-modules we have the following



When X is a scheme we get



and when X is a locally Noetherian scheme (sufficient for $\mathcal{O}_X$ to be coherent) we get



but this is sort of a "normal" place to work, in a non-technical sense most rings are Noetherian. So for normal algebraic geometry the category of coherent sheaves is a usefull abelian enlargment of the category of vector bundles. Aparently number theorists are not always in the nice Noetherian setting in which case the abelian enlargment will be quasi-coherent sheaves.

References

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes: With Examples and Exercises*. Springer Studium Mathematik - Master. Springer Fachmedien, Wiesbaden, 2020.

[Ris] The Rising Sea: Foundations of Algebraic Geometry. https://www.goodreads.com/book/show/53368091-the-rising-sea.